Quantum conductance problems and the Jacobi ensemble

P.J. Forrester
Department of Mathematics and Statistics,
University of Melbourne, Victoria 3010, Australia

In one dimensional transport problems the scattering matrix S is decomposed into a block structure corresponding to reflection and transmission matrices at the two ends. For S a random unitary matrix, the singular value probability distribution function of these blocks is calculated. The same is done when S is constrained to be symmetric, or to be self dual quaternion real, or when S has real elements, or has real quaternion elements. Three methods are used: metric forms; a variant of the Ingham-Seigel matrix integral; and a theorem specifying the Jacobi random matrix ensemble in terms of Wishart distributed matrices.

1 Introduction

In mesoscopic physics, the conductance of certain quasi one-dimensional wires containing scattering impurities exhibit the phenomenum known as universal conductance fluctuations. As first predicted theoretically [1, 11], and soon after observed experimentally [17], for such wires the variance of the conductance is of order $(e^2/h)^2$, independent of sample size or disorder strength. Furthermore the variance decreases by precisely a factor of two if time reversal symmetry is broken by a magnetic field. This conductance problem is fundamental for its relation to the Landauer scattering theory of electronic conduction (see [4] and references therein), and to time reversal symmetry. It is further fundamental for its relation to random matrix theory. Let us revise these points by way of background and motivation for the specific problem of this paper.

In the theoretical description of the conductance problem, basic quantities are the electron fluxes at the left and right hand edges of the wire. These are specified by an n-component vector $\vec{I}$ and an m-component vector $\vec{I}'$ specifying the complex amplitudes of the available plane wave states travelling into the left and right sides of the wire respectively, as well as an n-component vector $\vec{O}$ and an m-component vector $\vec{O}'$ for the same states travelling out of the left and right sides of the wire. For definiteness it will be assumed that $n \geq m$.

By definition, the scattering matrix S relates the flux travelling into the conductor to that travelling out,

$$S \begin{bmatrix} \vec{I} \\ \vec{I}' \end{bmatrix} := \begin{bmatrix} \vec{O} \\ \vec{O}' \end{bmatrix}. \tag{1.1}$$

The scattering matrix is further decomposed in terms of reflection and transmission matrices by

$$S = \begin{bmatrix} r_{n\times n} & t'_{n\times m} \\ t_{m\times n} & r'_{m\times m} \end{bmatrix}. \tag{1.2}$$

Flux conservation requires

$$|\vec{I}|^2 + |\vec{I'}|^2 = |\vec{O}|^2 + |\vec{O'}|^2,$$

and this implies that S must be unitary. Furthermore, by relating S to an evolution operator and thus a Hamiltonian, one can argue (see e.g. [6]) that S must be symmetric if the system has a time reversal symmetry with $T^2 = 1$, and a self dual quanternion matrix when there is a time reversal symmetry with $T^2 = -1$.

The immediate relevance of the above formalism is seen by invoking the Landauer scattering theory of electronic conduction. According to this formalism, the conductance G is given in terms of the transmission matrix $t_{m\times n}$ (or $t'_{n\times m}$) by the so called two probe Landauer formula

$$G/G_0 = \mathrm{Tr}(t^\dagger t) = \mathrm{Tr}(t'^\dagger t') \tag{1.3}$$

where $G_0 = 2e^2/h$ is twice the fundamental quantum unit of conductance. Thus to compute G it suffices to know the distribution of the eigenvalues of $t^\dagger t$ (or $t'^\dagger t'$). In fact the matrix S can be decomposed in a form which isolates these eigenvalues.

For definiteness suppose there is no time reversal symmetry, so S is a general unitary matrix. Each block of S can then be decomposed according to a general singular value decomposition. For example, for the block $r_{n\times n}$ we have

$$r_{n\times n} = U_r \Lambda_r V_r^\dagger,$$

where U_r, V_r are unitary matrices and Λ_r is a rectangular diagonal matrix with entries equal to the positive square roots of the eigenvalues of $r^\dagger r$ (these eigenvalues are between 0 and 1 since $r^\dagger r + t^\dagger t = I_{n\times n}$). The unitarity of S inter-relates the matrices $U_r, U_{r'}, \dots$ (see e.g. [16]) and implies the decomposition

$$S = \begin{bmatrix} U_r & 0 \\ 0 & U_{r'} \end{bmatrix} L \begin{bmatrix} V_r^\dagger & 0 \\ 0 & V_{r'}^\dagger \end{bmatrix} \tag{1.4}$$

where

$$L := \begin{bmatrix} \sqrt{1-\Lambda_t\Lambda_t^T} & i\Lambda_t \\ i\Lambda_t^T & \sqrt{1-\Lambda_t^T\Lambda_t} \end{bmatrix}. \tag{1.5}$$

Symmetries relate $V_r, V_{r'}$ to $U_r, U_{r'}$. As already remarked, in the case of a time reversal symmetry $T^2 = 1$, S must be symmetric, while for $T^2 = -1$, S must be self dual quaternion real. These symmetries require that

$$V_r^\dagger = U_r^T,\ V_{r'}^\dagger = U_{r'}^T \qquad \text{and} \qquad V_r^\dagger = U_r^D,\ V_{r'}^\dagger = U_{r'}^D \tag{1.6}$$

respectively. In (1.6) the operation D, for an $n\times n$ matrix A with real quaternion elements regarded as a $2n\times 2n$ matrix with complex elements, is specified by

$$A^D = Z_{2n} A^T Z_{2n}^{-1}, \qquad Z_{2n} := I_n \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The relevance of random matrix theory comes from hypothesizing that apart from the constraint imposed by a time reversal symmetry, in the regime relevant to universal conductance fluctuations, the scattering matrix S is effectively a random matrix chosen with Haar (uniform) measure. With this assumption the non-zero elements of Λ_t, which in turn are equal to the square root of the non-zero eigenvalues of $t^\dagger t$ and are thus in the interval $(0,1)$, has a joint probability density function (p.d.f.) proportional to

$$\prod_{j=1}^m \lambda_j^{\beta\alpha} \prod_{1\le j<k\le m} |\lambda_k^2 - \lambda_j^2|^\beta, \qquad \alpha := n - m + 1 - 1/\beta, \tag{1.7}$$

where $\beta = 1, 2, 4$ according to S being constrained to be symmetric, no constraints, or constrained to be self dual quanternion. In its full generality this result is stated without proof in Beenakker [4, eq. (2.9)], where it is attributed to Brouwer. In the case $n = m$ it was derived in [2, 14]. With knowledge of (1.7), it can be deduced from (1.3) that with α fixed [3]

$$\lim_{m,n\to\infty} \mathrm{Var}(G/G_0) = \frac{1}{8\beta}, \tag{1.8}$$

which is in quantitative agreement with experiment.

However the issue of fluctuation formulas is not our concern here. Rather we seek to specify three distinct derivations of (1.7) in the case $\beta = 2$, each of which we believe to have particular features of interest. The first derivation to be given uses the method of metric forms. This method applies to all three cases, and in fact details will be given in the case $\beta = 1$. The second derivation uses versions of the Ingham-Seigel integral; while the third relies on knowledge of the Jacobi random matrix ensemble as derived from Wishart distributed matrices. These latter two derivations generalize (1.7) in the case $\beta = 2$, and also allow β generalizations relating to the block decomposition of unitary matrices, however now with $\beta = 1$ refering to the elements of S being real, and $\beta = 4$ to the elements being real quaternion. Some aspects of statistical properties of the block decomposition of real orthogonal matrices (i.e. unitary matrices with real elements) have been previously given in [18].

2 Metric forms

Let $X = [x_{jk}]_{j,k=1,\dots,N}$ be an $N \times N$ matrix. Let $\{x_\mu\}$, where μ labels both rows and columns, be the set of independent real and imaginary parts in X. The metric form of the line element ds is defined by

$$(ds)^2 := \mathrm{Tr}(dX dX^\dagger) = \sum_{\mu \text{ diag.}} (dx_\mu)^2 + 2 \sum_{\substack{\mu \text{ upper} \\ \text{triangular}}} (dx_\mu)^2.$$

The corresponding volume measure is

$$(dX) = \bigwedge_{\mu \text{ diag.}} dx_\mu \bigwedge_{\substack{\mu \text{ upper} \\ \text{triangular}}} dx_\mu. \tag{2.1}$$

Suppose now a change of variables $\{x_\mu\} \mapsto \{y_\mu\}$ is made such that $(ds)^2$ is a symmetric quadratic form in $\{dy_\mu\}$,

$$(ds)^2 = \sum_{\mu,\nu} g_{\mu,\nu} dy_\mu dy_\nu, \qquad g_{\mu,\nu} = g_{\nu,\mu}.$$

The corresponding volume measure is then (see e.g. [12])

$$(\det[g_{\mu,\nu}])^{1/2} \bigwedge_\mu dy_\mu. \tag{2.2}$$

In [6] this formalism has been used to derive the result (1.7) in the case $\beta = 2$. Here the details will be given for the case $\beta = 1$, when (1.4) reads

$$S = \begin{bmatrix} U_r & 0 \\ 0 & U_{r'} \end{bmatrix} L \begin{bmatrix} U_r^T & 0 \\ 0 & U_{r'}^T \end{bmatrix}, \tag{2.3}$$

will be given.

For a general matrix A, let $\delta A := A^\dagger dA$, where dA denotes the matrix of differentials of the elements of A. Using the fact that for A unitary $(\delta A)^\dagger = -\delta A$, it follows from (2.3) that

$$\begin{bmatrix} U_r^\dagger & 0 \\ 0 & U_{r'}^\dagger \end{bmatrix} dS \begin{bmatrix} \bar{U}_r & 0 \\ 0 & \bar{U}_{r'} \end{bmatrix} = \begin{bmatrix} \delta U_r & 0 \\ 0 & \delta U_{r'} \end{bmatrix} L + dL - L \begin{bmatrix} \delta \bar{U}_r & 0 \\ 0 & \delta \bar{U}_{r'} \end{bmatrix}. \tag{2.4}$$

In $\mathrm{Tr}(dSdS^\dagger)$, the right hand side of (2.4) can effectively be substituted for dS. Doing this and simplifying using

$$\mathrm{Tr}\left(\begin{bmatrix} \delta U_r & 0 \\ 0 & \delta U_{r'} \end{bmatrix} L^\dagger dL \right) = 0$$

gives

$$\mathrm{Tr}(dSdS^\dagger) = \mathrm{Tr}\, A_1 \bar{A}_1 + \mathrm{Tr}(A_2 + \bar{A}_2) + \mathrm{Tr}(A_3 \bar{A}_3) + \mathrm{Tr}(dLdL^\dagger) \tag{2.5}$$

where, with I denoting the identity matrix,

$$\begin{aligned} A_1 &= \sqrt{I - \Lambda_t \Lambda_t^T}\, \delta \bar{U}_r - \delta U_r \sqrt{I - \Lambda_t \Lambda_t^T} \\ A_2 &= (\Lambda_t \delta \bar{U}_{r'} - \delta U_r \Lambda_t)(\Lambda_t^T \delta U_r - \delta \bar{U}_{r'} \Lambda_t^T) \\ A_3 &= \sqrt{I - \Lambda_t \Lambda_t^T}\, \delta \bar{U}_{r'} - \delta U_{r'} \sqrt{I - \Lambda_t \Lambda_t^T}. \end{aligned}$$

We recall that the diagonal elements of Λ_t are the positive square roots of the eigenvalues of $t^\dagger t$. Because $t = t_{m\times n}$ and thus has rank m, $t^\dagger t$ must have $n-m$ zero eigenvalues, so with the diagonal elements of $\Lambda_t \Lambda_t^T$ denoted λ_j ($j = 1, \dots, n$), as is consistent with the notation of (1.7), we have $\lambda_{m+1} = \cdots = \lambda_n = 0$. Using this fact we see that in component form the four terms in (2.5), $T_1, \dots, T_4$ say, can be expanded to read

$$\begin{aligned} T_1 &= \sum_{k=1}^{n} (1-\lambda_k^2) |(\delta \bar{U}_r)_{kk} - (\delta U_r)_{kk}|^2 + \sum_{1\le k<l\le n} \frac{1}{2}\Big(\sqrt{1-\lambda_l^2} + \sqrt{1-\lambda_k^2}\Big)^2 |(\delta \bar{U}_r)_{lk} - (\delta U_r)_{lk}|^2 \\ &\quad + \Big(\sum_{1\le k<l\le m} + \sum_{k=1}^{m} \sum_{l=m+1}^{n} \Big) \frac{1}{2}\Big(\sqrt{1-\lambda_l^2} - \sqrt{1-\lambda_k^2}\Big)^2 |(\delta \bar{U}_r)_{lk} + (\delta U_r)_{lk}|^2 \\ T_2 &= 2\Big(\sum_{k=1}^{m} \lambda_k^2 |(\delta \bar{U}_{r'})_{kk} - (\delta U_r)_{kk}|^2 + \sum_{1\le k<l\le m} \Big\{ \frac{1}{2}(\lambda_l + \lambda_k)^2 |(\delta \bar{U}_{r'})_{lk} - (\delta U_r)_{lk}|^2 \\ &\quad + \frac{1}{2}(\lambda_l - \lambda_k)^2 |(\delta \bar{U}_{r'})_{lk} + (\delta U_r)_{lk}|^2 \Big\} + \sum_{k=1}^{m} \sum_{l=m+1}^{n} \lambda_k^2 |(\delta U_r)_{lk}|^2 \Big) \\ T_3 &= \sum_{k=1}^{m} (1-\lambda_k^2) |(\delta \bar{U}_{r'})_{kk} - (\delta U_{r'})_{kk}|^2 + \sum_{1\le k<l\le m} \frac{1}{2}\Big(\sqrt{1-\lambda_l^2} + \sqrt{1-\lambda_k^2}\Big)^2 |(\delta \bar{U}_{r'})_{lk} - (\delta U_{r'})_{lk}|^2 \\ &\quad + \frac{1}{2}\Big(\sqrt{1-\lambda_l^2} - \sqrt{1-\lambda_k^2}\Big)^2 |(\delta \bar{U}_{r'})_{lk} + (\delta U_{r'})_{lk}|^2 \\ T_4 &= 2 \sum_{j=1}^{m} \frac{(d\lambda_j)^2}{1-\lambda_j^2}. \end{aligned}$$

In general a symmetric unitary matrix has the same number of independent elements as a real symmetric matrix of the same rank, so S has $\frac{1}{2}(m+n)(m+n+1)$ independent elements. We thus seek this same number of independent differentials in $T_1, \dots, T_4$. These are

$$(\delta U_r)_{kk}^{(\mathrm{i})}, \quad 1 \le k \le n \qquad (\delta U_{r'})_{kk}^{(\mathrm{i})}, \quad 1 \le k \le m$$

and

$$(\delta U_r)^{(\mathrm{r})}_{lk}, \quad (\delta U_r)^{(\mathrm{i})}_{lk}, \quad (\delta U_r)^{(\mathrm{r})}_{lk}, \quad (\delta U_r)^{(\mathrm{i})}_{lk}, \quad 1 \le k < l \le m,$$

as well as

$$(\delta U_r)^{(\mathrm{r})}_{lk}, \quad (\delta U_r)^{(\mathrm{i})}_{lk}, \qquad 1 \le k \le m \,\&\, m+1 \le l \le n$$

and

$$(\delta U_r)^{(\mathrm{i})}_{lk}, \quad m+1 \le l < k \le n \qquad d\lambda_j \quad 1 \le j \le m,$$

which indeed tally to $\frac{1}{2}(m+n)(m+n+1)$.

We see from the expressions for T_1, T_2, T_3 that the contribution to the metric form from the differentials with subscripts lk such that $1 \le k < l \le m$ is

$$\begin{aligned} 4 \sum_{1 \le k < l \le m} & \Big(1 + \sqrt{1-\lambda_l^2}\sqrt{1-\lambda_k^2}\Big)\Big((\delta U_r^{(\mathrm{i})})^2_{lk} + (\delta U_{r'}^{(\mathrm{i})})^2_{lk}\Big) \\ & + \Big(1 - \sqrt{1-\lambda_l^2}\sqrt{1-\lambda_k^2}\Big)\Big((\delta U_r^{(\mathrm{r})})^2_{lk} + (\delta U_{r'}^{(\mathrm{r})})^2_{lk}\Big) \\ & -2\lambda_l\lambda_k(\delta U_{r'}^{(\mathrm{r})})_{lk}(\delta U_r^{(\mathrm{r})})_{lk} + 2\lambda_l\lambda_k(\delta U_{r'}^{(\mathrm{i})})_{lk}(\delta U_r^{(\mathrm{i})})_{lk}. \end{aligned} \tag{2.6}$$

Setting $\sqrt{1-\lambda_l^2}\sqrt{1-\lambda_k^2} =: a$ for notational convenience, this portion of the metric form contributes to $(\det[g_{jk}])^{1/2}$ in (2.2) 2×2 block factors which is proportional to

$$\prod_{k<l}^{m} \begin{vmatrix} 1+a & -\lambda_l\lambda_k \\ -\lambda_l\lambda_k & 1+a \end{vmatrix}^{1/2} \begin{vmatrix} 1-a & \lambda_l\lambda_k \\ \lambda_l\lambda_k & 1-a \end{vmatrix}^{1/2} = \prod_{k<l}^{m} |\lambda_l^2 - \lambda_k^2|. \tag{2.7}$$

For $m+1 \le k < l \le n$, the coefficient of $(\delta U_r)^{(\mathrm{i})}_{lk}$ in T_1 is independent of the λ's and so for the present purposes can be ignored. For $1 \le k \le m$, $m+1 \le l \le n$, we see from T_1 and T_2 that the corresponding contribution to the metric form is

$$4 \sum_{k=1}^{m} \sum_{l=m+1}^{n} (1 + \sqrt{1-\lambda_k^2})((\delta U_r)^{(\mathrm{i})}_{lk})^2 + (1 - \sqrt{1-\lambda_k^2})((\delta U_r)^{(\mathrm{r})}_{lk})^2.$$

This contributes to the volume form a factor proportional to

$$\prod_{k=1}^{m} \prod_{l=m+1}^{n} \lambda_k = \prod_{k=1}^{m} \lambda_k^{n-m}. \tag{2.8}$$

We read off that the contribution to $(\det[g_{jk}])^{1/2}$ from the coefficients of the terms $(d\lambda_j)^2$ in T_4 is proportional to

$$\prod_{k=1}^{m} \frac{1}{(1-\lambda_k^2)^{1/2}}. \tag{2.9}$$

It remains to calculate the contribution from the differentials on the diagonal. For $m+1 \le k \le n$, the coefficient of $(\delta U_r)^{(i)}_{kk}$ is a constant so these differentials can be ignored. We read off from T_1, T_2, T_3 that the contribution to the metric form from the remaining differentials is

$$2 \sum_{k=1}^{m} (2-\lambda_k^2)\Big(((\delta U_r)^{(\mathrm{i})}_{kk})^2 + (\delta U_{r'})^{(\mathrm{i})}_{kk})^2\Big) + 2\lambda_k^2 (\delta U_r)^{(\mathrm{i})}_{kk}(\delta U_{r'})^{(\mathrm{i})}_{kk}.$$

The contribution to the volume form is thus proportional to

$$\prod_{k=1}^{m} \begin{vmatrix} 2-\lambda_k^2 & \lambda_k^2 \\ \lambda_k^2 & 2-\lambda_k^2 \end{vmatrix}^{1/2} \propto \prod_{k=1}^{m} (1-\lambda_k^2)^{1/2}. \tag{2.10}$$

Multiplying together (2.7)–(2.10) gives (1.7) in the case $\beta = 1$.

3 Matrix integrals

Let

$$I^{(2)}_{m,n}(Q_m) := \int e^{\frac{i}{2}\mathrm{Tr}(H_m Q_m)} \Big(\det(H_m - \mu I_m) \Big)^{-n} (dH_m) \tag{3.1}$$

where H_m, Q_m are $m \times m$ Hermitian matrices, and suppose $n \ge m$, $\mathrm{Im}\,\mu > 0$. For Q_m positive definite, it has been proved by Fyodorov [7] that

$$I^{(2)}_{m,n}(Q_m) = \frac{2^m \pi^{m(m+1)/2} i^m (-1)^{m(m-1)/2}}{\prod_{j=0}^{m-1}\Gamma(n-j)} \Big(\det(\frac{i}{2}Q_m) \Big)^{n-m} e^{\frac{i}{2}\mu \mathrm{Tr}\, Q_m}. \tag{3.2}$$

(Here we have taken $(dH_m) := \prod_{j=1}^m dh^{(\mathrm{i})}_{jj} \prod_{j<k} dh^{(\mathrm{r})}_{jk} dh^{(\mathrm{i})}_{jk}$ which differs by a factor of 2 in the product over $j<k$ to the convention adopted in [7].) This matrix integral may be regarded as being of the type first evaluated by Ingham and Siegel (see [7] and references therein). In a subsequent work [8], (3.1) was used to derive (1.7) in the case $\beta = 2$. Here we will show that this derivation can be used to derive a generalization of (1.7) in the case $\beta = 2$, and this generalization can be further extended to the cases $\beta = 1$ and $\beta = 4$, using suitable variants of (3.1), where now $\beta = 1$ and 4 refers to the decomposition (1.2) with S having real and real quaternion elements respectively.

Define by $I^{(1)}_{m,n}(Q_m)$ the matrix integral (3.1) with H_m and Q_m now real symmetric. Also, define by $I^{(4)}_{m,n}(Q_m)$ the same matrix integral but with H_m and Q_m now self dual quaterion Hermitian matrices (such matrices regarded as $2m \times 2m$ Hermitian matrices are doubly degenerate; adopt the convention that the operations Tr and det include only distinct eigenvalues). In [7] it is remarked that the method of derivation given therein to deduce that the evaluation of $I^{(2)}_{m,n}(Q_m)$ can also be used to deduce the evaluation of $I^{(1)}_{m,n}(Q_m)$, which reads

$$I^{(1)}_{m,n/2}(Q_m) = \frac{2^m \pi^{m(m+3)/2} i^{m(m+1)/2}}{\prod_{j=0}^{m-1}\Gamma((n-j)/2)} \Big(\det(\frac{i}{2}Q_m) \Big)^{(n-m-1)/2} e^{\frac{i}{2}\mu \mathrm{Tr}\, Q_m}. \tag{3.3}$$

Applying the same method to $I^{(4)}_{m,n}(Q_m)$ gives

$$I^{(4)}_{m,2n}(Q_m) = \frac{(2i)^m \pi^{m^2}}{\prod_{j=0}^{m-1}\Gamma(2(n-j))} \Big(\det(\frac{i}{2}Q_m) \Big)^{2(n-m+1/2)} e^{\frac{i}{2}\mu \mathrm{Tr}\, Q_m}. \tag{3.4}$$

Hence, all three cases we have

$$I^{(\beta)}_{m,\beta n/2}(Q_m) = C^{(\beta)}_{m,n} (\det(Q_m))^{(\beta/2)(n-m+1-2/\beta)} e^{\frac{i}{2}\mu \mathrm{Tr}\, Q_m}, \tag{3.5}$$

where $C^{(\beta)}_{m,n}$ is independent of Q_m.

Let U be an $N \times N$ random unitary matrix, with real ($\beta = 1$), complex ($\beta = 2$) and real quaternion ($\beta = 4$), chosen with Haar measure. Generalizing (1.2), decompose U into blocks

$$U = \begin{bmatrix} A_{n_1 \times n_2} & C_{n_1 \times (N-n_2)} \\ B_{(N-n_1)\times n_2} & D_{(N-n_1)\times(N-n_2)} \end{bmatrix}. \tag{3.6}$$

Since U is unitary, we require

$$AA^\dagger + CC^\dagger = I_{n_1}, \tag{3.7}$$

together with three similar equations involving B and D which given A and C can always be satisfied. The idea of [18] is to regard (3.7) as a constraint in the space of general rectangular matrices A, C with entries of the type required by the index β. Thus in this viewpoint the distribution of A is given by

$$\int \delta(AA^\dagger + CC^\dagger - I_{n_2})(dC). \tag{3.8}$$

In (3.8) the delta function is a product of scalar delta functions, one for each independent real and imaginary component of $A^\dagger A + C^\dagger C - I_{n_2}$. It is proportional to the matrix integral

$$\int e^{-i\mathrm{Tr}(H(AA^\dagger + CC^\dagger - I_{n_2}))}(dH) \tag{3.9}$$

where H is an Hermitian matrix with elements of the type β. Following [8] we would like to substitute (3.9) for the delta function in (3.8), and change the order of integration. The integrations over (dC) are simply Gaussian integrals. For the resulting function of H to be integrable around $H = 0$, the replacement $H \mapsto H - i\mu I_{n_1}$ in the exponent of (3.9) must made. Doing this and computing the Gaussian integrals gives that (3.8) is proportional to

$$\lim_{\mu\to 0^+} \int \Big(\det(H - i\mu I_{n_1}) \Big)^{-\beta(N-n_2)/2} e^{i\mathrm{Tr}(H(I_{n_1} - AA^\dagger))}(dH)$$

Evaluating the matrix integral using (3.5) shows that the distribution of A is proportional to

$$\Big(\det(I_{n_1} - AA^\dagger) \Big)^{(\beta/2)(N-n_1-n_2+1-2/\beta))}. \tag{3.10}$$

Note that for this to be normalizable, we must have

$$N - n_1 - n_2 \geq 0. \tag{3.11}$$

Suppose that in addition to (3.11) we have $n_1 \geq n_2$. Then A has rank n_1 and so $AA^\dagger$ has $n_1 - n_2$ zero eigenvalues and (3.10) can be written

$$\Big(\det(I_{n_2} - A^\dagger A) \Big)^{(\beta/2)(N-n_1-n_2+1-2/\beta)}. \tag{3.12}$$

Setting $Y = A^\dagger A$ we know (see e.g. [6]) that

$$dA \propto (\det Y)^{(\beta/2)(n_1-n_2+1-2/\beta)}(dY), \tag{3.13}$$

so we have from (3.12) that the distribution of Y is proportional to

$$(\det Y)^{(\beta/2)(n_1-n_2+1-2/\beta))} \Big(\det(I_{n_2} - Y) \Big)^{(\beta/2)(N-n_1-n_2+1-2/\beta)}. \tag{3.14}$$

Denote the eigenvalues of Y by $y_1, \dots, y_{n_2}$. Using the fact that the eigenvalue dependent portion of the Jacobian for an Hermitian matrix, with elements of the type β, when changing variables to its eigenvalues and eigenvectors is $\prod_{j<k} |y_k - y_j|^\beta$, we read off from (3.14) that the eigenvalue p.d.f. of Y is proportional to

$$\prod_{j=1}^{n_2} y_j^{(\beta/2)(n_1-n_2+1-2/\beta)} (1-y_j)^{(\beta/2)(N-n_1-n_2+1-2/\beta))} \prod_{j<k}^{n_2} |y_k - y_j|^\beta. \tag{3.15}$$

In the case

$$n_1 = n_2 = m, \qquad N = n + m, \tag{3.16}$$

the matrix A in (3.6) coincides with r in (1.2), and the y_j in (3.15) are then the eigenvalues of $r^\dagger r$. The non-zero values of the singular values of the submatrix t in (1.2) are the λ_j's in (1.7). The decomposition (1.4) tells us that $y_j = 1 - \lambda_j^2$. Indeed, making this change of variable in (3.15), and making the substitutions (3.16), reclaims (1.7) in the case $\beta = 2$.

For $\beta = 2$ and general $n_1 \geq n_2$, $N \geq n_1 + n_2$, a result equivalent to (3.15) is derived in a recent work of Simon and Moustakas [15]. Motivated by a quantum dot problem with three leads, they decomposed the $N \times N$ scattering matrix S into a 3×3 block structure

$$S = \begin{bmatrix} r_{11} & t_{12} & t_{13} \\ t_{21} & r_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \tag{3.17}$$

where each r_{ii} is $N_i \times N_i$ and t_{ij} is $N_i \times N_j$, with $N = N_1 + N_2 + N_3$. For S a random unitary matrix with Haar measure, it is shown that the eigenvalues of $t_{12}^\dagger t_{12}$ have p.d.f. given by (3.15) with $\beta = 2$, $n_1 = N$, $n_2 = N_2$. This is consistent with our result because the distribution of S is unchanged by interchanging rows and columns. The first and second block columns in (3.17) can be interchanged, effectively giving the decomposition (3.6) with $t_{12} = A_{N_1 \times N_2}$.

We remark that for $y_j \ll 1$ the term involving $(1 - y_j)$ in (3.15) can be ignored. The resulting p.d.f. is then the $x_j \ll 1$ limit of the singular values p.d.f. of the matrix product $X^\dagger X$, where X is an $n_1 \times n_2$ random Gaussian matrix with real ($\beta = 1$), complex ($\beta = 2$) or real quanternion elements ($\beta = 4$) (see e.g. [6]). This is consistent with a recent result of Jiang [10], who quantifies the degree to which the entries of orthogonal or unitary random matrices can be approximated by entries of Gaussian random matrices with real or complex entries respectively.

4 Random projections

In a recent work Collins [5] computed the distribution of certain products of random orthogonal projections. Here this approach, suitably modified, will be used to derive (3.15). Knowledge is required of the following result [13, 6]

Proposition 4.1. *Let c and d be $n_1 \times m$ and $n_2 \times m$ Gaussian random matrices, all elements identically and independently distributed (i.i.d.) where $n_2, n_2 \geq m$, indexed by the parameter $\beta = 1, 2$ or 4. The parameter β specifies that the elements are real ($\beta = 1$), complex ($\beta = 2$) or real quaternion ($\beta = 4$). Let $C = c^\dagger c$, $D = d^\dagger d$. The distribution of $J := (C + D)^{-1/2} C (C + D)^{-1/2}$ is proportional to*

$$(\det J)^{a\beta/2} (\det(I - J))^{b\beta/2} \tag{4.1}$$

with

$$a = n_1 - m + 1 - 2/\beta, \qquad b = n_2 - m + 1 - 2/\beta.$$

The p.d.f. (4.1) is referred to as the Jacobi ensemble (see e.g. [6]). To make use of Proposition 4.1, let X be a $N \times m$, $N \geq m$ i.i.d. Gaussian random matrix indexed by β. Define Q_1 to be the top $n \times N$ submatrix of an $N \times N$ unitary matrix indexed by β. Because $Q_1 Q_1^\dagger = I_n$, we have that $Q_1 X$ is distributed as an $n \times m$ i.i.d. Gaussian random matrix indexed by β (see [9, Thm. 2.3.1]). Let Q_2 be the bottom $(N - n) \times N$ submatrix of U so that

$$Q_1^\dagger Q_1 + Q_2^\dagger Q_2 = I_N. \tag{4.2}$$

Because $Q_2 Q_2^\dagger = I_{N-n}$, $Q_2 X$ is distributed as an $(N - n) \times m$ i.i.d. Gaussian matrix indexed by β.

Now set

$$Y = X^\dagger Q_1^\dagger Q_1 X, \qquad Z = X^\dagger Q_2^\dagger Q_2 X. \tag{4.3}$$

Denoting by $W_p^{(\beta)}(n)$ the (Wishart) distribution of the matrix product $A^\dagger A$, where A is an $n \times p$, $(n \geq p)$ i.i.d. Gaussian random matrix indexed by β, we see that the distribution of Y is $W_m^{(\beta)}(n)$ while the distribution of Z is $W_m^{(\beta)}(N-n)$. The result of Proposition 4.1 tells us that the eigenvalue p.d.f. of

$$(Y+Z)^{-1/2} Y (Y+Z)^{-1/2} = (X^\dagger X)^{-1/2} Y (X^\dagger X)^{-1/2} \tag{4.4}$$

is given by the Jacobi ensemble (4.1) with $n_1 = n$, $n_2 = N - n$.

According to the singular value decomposition, we can write

$$X = U_1 \Lambda U_2 \tag{4.5}$$

for U_1 an $N \times N$ unitary matrix, U_2 an $m \times m$ unitary matrix, and Λ an $N \times m$ diagonal matrix, with diagonal entries equal to the positive square root of the eigenvalues of $X^\dagger X$. It follows from (4.5) that

$$(X^\dagger X)^{1/2} = U_2 [\Lambda]_{m,m} U_2 \tag{4.6}$$

where $[\Lambda]_{m,m}$ refers to the top $m \times m$ sub-block of Λ, while (4.6) and (4.5) together imply

$$X(X^\dagger X)^{-1/2} = [U_1]_{N,m} U_2^\dagger. \tag{4.7}$$

Substituting for Y in (4.4) according to (4.3), then using (4.7) to substitute for $X(X^\dagger X)^{-1/2}$ tells us that the distribution of

$$U_2 [U_1]_{N,m}^\dagger Q_1^\dagger Q_1 [U_1]_{N,m} U_2^\dagger \tag{4.8}$$

is given by (4.1). We note

$$U_1^\dagger [U_1]_{N,m} = \mathrm{diag}(1, \dots, 1, 0, \dots, 0)$$

where there are m 1's. Since the distribution of Q_1 is unchanged upon multiplication by a unitary matrix, the distribution of (4.8) is the same as the distribution of

$$U_2 [Q_1^\dagger Q_1]_{m,m} U_2^\dagger. \tag{4.9}$$

In terms of the decomposition (3.6) this in turn has the same distribution as

$$A^\dagger A \Big|_{\substack{n_1 \mapsto n \\ n_2 \mapsto m}}. \tag{4.10}$$

Hence we have the distribution of (4.10) is given by (4.1) with

$$a = n - m + 1 - 2/\beta, \qquad b = N - n - m + 1 - 2/\beta,$$

which is in precise agreement with (3.14) upon the replacements $n_1 \mapsto n$, $n_2 \mapsto m$ in the latter.

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